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Contemporary methods for solving nonconvex problems of optimization and optimal control

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Аннотация. The paper suggests a survey of the results obtained by the author and his adherents in recent years and related to nonconvex optimization and nonconvex problems of optimal control. Statements of the related applied problmes and some results are given.

Ключевые слова: nonconvex optimization, difference of two convex functions, local search, global search.

1. Introduction

In recent three decades attention of the specialists involved in solving extremum problems has often been attracted to nonconvex problems of optimization [1, 2], which may have a sufficiently large number of local solutions and stationary (critical, say, in the sense of Ferma or Lagrange conditions) points, which are rather far from – even in the aspect of the goal functional – from the global solution, which is so essential from the practical viewpoint. Meanwhile, convex problems, as known from [3]-[5], possess the property that even each local solution turns out to be also global.

Noteworthy, in practice, convex problems occur rather seldom, while there are many examples of convex problems in textbooks [1]-[5]. Furthermore, in applied problems, nonconvex structures most frequently occur in a hidden form. As a consequence, the specialists not always pay attention to their presence and to the nature of their occurrence, as, for example, in problems of hierarchical optimization and optimal control of nonlinear systems. One knows that it is difficult to hope for any success without knowing the structure of the convexity. Note that rather frequently nonconvex structures are generated by convex ones (complementing of the convex set, maximization of the convex function, etc.)[1, 2, 6].

On the other hand, specialists in applied problems do not think about the correctness of direct application of classical methods of optimization in nonconvex problems, while the numerical results are interpreted only in the content aspect, without thinking of the fact that all classical optimization methods (Newtonian ones, methods of conjugate gradients, methods of feasible directions, barrier methods, etc.) converge to the global solution only in convex problems [7]-[6].

At the same time, in nonconvex problems, the direct application of standard methods may have unpredictable consequences [1]-[5], and sometimes may even distract one from the desired solution. So, it is quite natural (but hardly ever grounded) is the reaction of the specialists propagating methods of direct selection – such as the method of branches and bounds (and cuts methods), which, as known, suffer curse of dimension, when the volume of computations grows exponentially side by side with the growth of the problem's dimension [1, 2]. We are sure, there exists also another way of solving nonconvex problems of high dimension [6]-[24].

In the recent two decades, we have managed to construct a theory of global search, which is harmonic from the viewpoint of the theory of optimization and which unexpectedly has turned out to be rather efficient in the aspect of computations, especially for the problems of high dimensions. Necessary and sufficient Global Optimality Conditions (GOCs) for the principal classes of nonconvex problems have turned out to be the kernel of this theory (see below) [6].

On the other hand, we have proposed a family of local search methods (LSMs), which, on the one hand, in some cases develop methods earlier known for the special problems and, on the other hand, this family of LSms represents a joint ensemble of methods, which is harmonic from the viewpoint of GOCs [6, 7, 10, 11].

Meanwhile, the procedures of escape from the stationary or local solutions, which are based on GOCs, are unique and quite efficient even in case of any simplest implementation [6, 7].

The approach elaborated has been tested on a wide field of nonconvex problems (some part of which is represented below). It has demonstrated an unexpected efficiency during the numerical solving problems of high dimension. Note, convex optimization methods are successfully used "inside" the procedures of local and global search proposed [6]-[23].

2. Classification

On the present stage, the class of d.c. functions $DC(I\!R^n)$, which may be represented in the form of difference of two convex functions

$$f(x) = g(x) - h(x),\ x \in I\!R^n,\ g, h \in CONV(I\!R^n). \tag{2.1}$$

is considered to be rather wide for the consideration. This class possess several remarkable properties.

a) Set $DC(I\!R^n)$ has been generated by the well-studied class — the cone of convex functions and represents a linear space [1, 6].

b) $DC(I\!R^n)$ includes the well-known classes such as dually differentiable functions, power and trigonometric polynomials, I.I.Eremin's functions, etc. [1, 6].

c) An arbitrary continuous function on a compact $K \subset I\!R^n$ may be arbitrarily approximated (in the topology of homogeneuos convergence) by a function from $DC(K)$ [1]-[6]. Consequently, any continuous problem of optimization on a compact may be approximated by a problem of optimization with d.c. functions. Anyway, note, if f represents a d.c. function, then there exists an infinite number of d.c. representations of the form (2.1), for example, of the form of difference of strongly convex functions.

Closeness of the DC class with respect to the majority of operations, which are used in optimization, is also an essential property of this DC class, which is important from the optimization viewpoint. For example, a sum, a difference, the module, the maximum, the minimum, etc. of the d.c. functions occur also in the class $DC(I\!R^n)$.

At the same time, the number of problems with d.c. functions is so large that the majority of the specialists, who have a long-time experience of solving problems of d.c. programming are sure [1, 2] that all (or almost all) nonconvex optimization problems are d.c. problems. In this connection, the following classification of d.c. programming problems may be considered to be natural.

1. **D.C. minimization**

$$(\mathcal{P}) \qquad f(x) = g(x) - h(x) \downarrow \min, \quad x \in D, \tag{2.2}$$

where $g(\cdot)$, $h(\cdot)$ are convex functions, and D is a convex set, which may be given by either inequalities or by equalities.

2. **Problems with d.c. constraints**, which are reducible to the following problem

$$\left.\begin{array}{l} f_0(x) \downarrow \min,\ x \in S, \\ f(x) = g(x) - h(x) \le 0, \end{array}\right\} \tag{2.3}$$

where $g(\cdot)$, $h(\cdot)$ are convex functions, $S \subset I\!R^n$, $f_0(\cdot)$ is a continuous function.

Particular cases of these problems are:

3. **Convex maximization**

$$h(x) \uparrow \max, \quad x \in D, \tag{2.4}$$

(when $g \equiv 0$ in (2.2)).

4. **Problems with inverse-convex constraints**

$$\left.\begin{array}{l} f_0(x) \downarrow \min,\ x \in S, \\ h(x) \ge 0, \end{array}\right\} \tag{2.5}$$

($g \equiv 0$ in (2.3)).

Note, any quadratic problem of optimization with sign-definite matrices occurs in this classification.

3. Local search

Unlike that in known procedures (such as procedures of "branches and boundaries" , cuts, etc.), which, say, "have distracted", as known, contemporary methods of convex optimization, we insist on the obligatory but "indirect" application of these methods in global optimization [3, 5]. For example, as regards the problem of d.c. minimization ($\mathcal{P}$)–(2.2), the basic element, the "corner stone" — in our opinion — is solving the following convex problem (linearized at a current point $x^s \in D$)

$$(\mathcal{PL}_s) \qquad g(x) - \langle \nabla h(x^s), x \rangle \downarrow \min \; x \in D. \tag{3.1}$$

Depending on the choice of the method of solving this problem ("the corner stone"), the global search (the total "building") may turn out to be either successful or not: efficient or not, "stable" with respect to the choice of the initial approximation or capable of obtaining – in the best case – simply a feasible point.

The local search itself (Local Search Procedure, LSP) may imply, for example, the sequential (likewise in the method of direct iteration) solving the problems ($\mathcal{PL}_s$) – (3.1). If we know $x^s \in D$ then we can find $x^{s+1} \in D$ in the capacity of an approximate solution ($\mathcal{PL}_s$). It is surprising that the process in this case converges (w.r.t. the function $f = g - h$) [6] to a solution x_* of the linearized problem

$$(\mathcal{PL}_*) \qquad g(x) - \langle \nabla h(x_*), x \rangle \downarrow \min \; x \in D,$$

(x_* is a critical point w.r.t. the method of local search). Under additional assumptions of strongly convex decomposition $f = g - h$ it is possible to provide for the convergence of $x^s \to x_*$.

Very frequently this method gives the global solution on small dimensions ($n \leq 7 - 10$), what adds to the set of difficulties related to constructing good ("bad") initial approximations for testing the global search.

Special methods of local search have been developed for the problems with d.c. constraints [11, 12]. These methods have also been grounded on considering linearized problems of the form:

$$(\mathcal{LP}_s) \qquad \left. \begin{array}{l} g(x) - \langle \nabla h(x^s), x \rangle \downarrow \min, \\ x \in S, \; f_0(x) \leq \xi_k. \end{array} \right\} \tag{3.2}$$

4. Global search

The general procedure of global search includes the two parts:

a) local search;

b) procedures of escape from a critical point, which is based on GOCs [6, 7], with subsequent inclusion of the procedure of local search.

The point is that GOCs possess a so called algorithmic (constructive) property, which allows one — in case of violation of these GOCs — to construct a feasible point, which is better than the point under scrutiny.

Indeed, for example, for GOCs for the problem $(\mathcal{P})$ – (2.2) have the following form [6, 7]:

$$z \in Sol(\mathcal{P}) \Rightarrow \ \forall (y, \beta) \in I\!R^n \times I\!R :$$

$$h(y) = \beta - \zeta, \ \zeta := f(z), \tag{4.1}$$

$$g(x) - \beta \geq \langle \nabla h(y), x - y \rangle \ \forall x \in D. \tag{4.2}$$

If for some $(\hat{y}, \hat{\beta})$ from (4.1) and $\hat{x} \in D$ (4.2) is violated

$$g(\hat{x}) < \hat{\beta} + \langle \nabla h(\hat{y}), \hat{x} - \hat{y} \rangle,$$

then the convexity of h implies that

$$f(\hat{x}) = g(\hat{x}) - h(\hat{x}) < h(\hat{y}) + \zeta - h(\hat{y}) = f(z),$$

or $f(\hat{x}) < f(z)$. So, $\hat{x} \in D$ is "better" than z. So, when selecting the "perturbation parameters" (y, β) in (4.1) and solving linearized problems (see (4.2)),

$$g(x) - \langle \nabla h(y), x \rangle \downarrow \min, \ x \in D, \tag{4.3}$$

(where y is not obligatorily feasible (!!!)), we obtain a family of initial points $x(y, \beta)$ for the LSP. Furthermore, on each level ζ it is not necessary to conduct selection inside all the set of (y, β) — it is sufficient to discover the violation of (4.2) only for one pair $(\hat{y}, \beta)$. After that, it is necessary to proceed to a new level $z^{k+1} := \hat{x}, \ \zeta_{k+1} := f(z^{k+1})$, and start the procedure from the very beginning.

A wide field of computational experiments has confirmed an unexpectedly high efficiency of the approach proposed, especially for the problems of high dimension, even in the case of program implementations conducted by students and postgraduates [7], [6]–[12].

5. Applied problems

5.1. Bimatrix games and bilinear programming

Bimatrix games reflect the conflict of the two parties, each one having a finite number of strategies. Some economics, engineering and ecological problems may be represented in the form of bimatrix games, in which the Nash equilibrium is the common concept. We have elaborated a new method for finding situations of Nash equilibrium in bimatrix games, which is based on the so called variational approach to solving game problems [7]. This means that the search for the Nash situation $(x^*, y^*) \in S_m \times S_n$, where $S_m,\ S_n$ are canonical symplexes, is reduced to solving the following nonconvex problem of mathematical programming:

$$\left.\begin{array}{c} F(x,y,\alpha,\beta) \overset{\triangle}{=} \langle x, (A+B)y\rangle - \alpha - \beta \uparrow \max, \\ x^T B - \beta e_n \leq 0_n,\ \ x \in S_m, \quad Ay - \alpha e_m \leq 0_m,\ \ y \in S_n, \end{array}\right\} \tag{5.1}$$

where $e_p = (1,1,...,1)^T \in I\!R^p$, S_p is the canonical symplex, $p = m, n$.

To the end of solving this problem, we have developed and grounded special algorithms of local and global search. The algorithms constructed have been tested on specially generated bimatrix games of high dimension (up to 1000×1000).

Furthermore, we have elaborated a parallel version of the algorithm of global search for Nash equilibria in the bimatrix games, which has allowed us to obtain an almost linear parallel increase of the rate of the process (by 5 to 6 times on 8 processors).

Next, the technique of global search in bimatrix games was generalized to be applied to the problems of bilinear programming [7] of the form:

$$\left.\begin{array}{c} F(x,y) = \langle c, x\rangle + \langle x, Cy\rangle + \langle d, y\rangle \uparrow \max\limits_{(x,y)}, \\ x \in X \overset{\triangle}{=} \{x \in I\!R^m \mid Ax \leq a,\ x \geq 0\}, \\ y \in Y \overset{\triangle}{=} \{y \in I\!R^n \mid By \leq b,\ y \geq 0\}. \end{array}\right\} \tag{BLP}$$

5.2. Hierarchical problems and variational inequalities

Hierarchical problems are encountered in practice because of impossibility of accumulation of the total available information at the upper level in the process of investigation of structurally complex control systems (social, economic, ecological-economic ones, etc.) and possess some hidden nonconvexity. In particular, problems of bilevel programming represent extremum problems, which – side by side with standard constraints, which are expressed in terms of equalities and inequalities, include the constraint described with the aid of an optimization subproblem representing the lowest level of the bilevel problem.

a) **Linear bilevel problems**

Consider the problem:

$$\left.\begin{array}{c} F(x,y) \overset{\triangle}{=} \langle c,x\rangle + \langle d,y\rangle \downarrow \min\limits_{x}, \\ (x,y) \in X \overset{\triangle}{=} \{x \in I\!R^m \mid Ax \le b\}, \\ y \in Y_*(x) \overset{\triangle}{=} \operatorname{Arg}\min\limits_{y} \left\{ \langle d^1, y\rangle \mid y \in Y(x) \overset{\triangle}{=} \{y \in I\!R^n \mid A_1x + B_1y \le b^1\} \right\} \end{array}\right\} (\mathcal{BP})$$

To the end of its solving we have proposed and tested the two approaches: i) an approach based on the application of the penalty method in combination with the global search strategy (GSS) in problems of d.c. minimization and ii) an GSS for the problems with d.c. constraints.

b) **Nonlinear bilevel problems**

Consider the following quadratic-linear problem of bilevel programming:

$$\left.\begin{array}{c} F(x,y) \overset{\triangle}{=} \frac{1}{2}\langle x, Cx\rangle + \langle c,x\rangle + \frac{1}{2}\langle y, C_1y\rangle + \langle c_1, y\rangle \downarrow \min\limits_{x,y}, \\ (x,y) \in X \overset{\triangle}{=} \{(x,y) \in I\!R^m \times I\!R^n \mid Ax + By \le a, \ \ x \ge 0\}, \\ y \in Y_*(x) \overset{\triangle}{=} \operatorname{Arg}\min\limits_{y}\{\langle d,y\rangle \mid \ y \in Y(x)\}, \\ Y(x) \overset{\triangle}{=} \{y \in I\!R^n \mid A_1x + B_1y \le b, \ \ y \ge 0\}. \end{array}\right\} (\mathcal{BP})$$

The following two types of solutions in such a problem are investigated: the optimistic one and the pessimistic (guaranteed) one.

We have elaborated a software complex intended for finding optimistic solutions in quadratic-linear problems of bilinear programming on the basis of the technique proposed earlier, which is based on the reduction of bilevel problems to a family of mathematical programming problems with a d.c. goal function. Testing of the software complex elaborated has been conducted on a large set of randomly generated problems of diverse complexity and dimension (up to 150×150). This testing has demonstrated that the methods proposed are rather efficient. Note, results of solving quadratic-linear problems of such a dimension cannot be found in the available literature.

5.3. Linear problem of complementarity

A similar variation approach has been employed for solving the well-known linear complementarity problem (LCP), which implies finding the pair of vectors (x, w), which satisfy the following conditions:

$$\left.\begin{array}{c} Mx + q = w, \ \ \langle x, w\rangle = 0, \\ x \ge 0, \ \ w \ge 0, \end{array}\right\} \tag{5.2}$$

where $x, w \in I\!R^n$, vector $q \in I\!R^n$ and the real sign-definite $(n \times n)$-matrix M are given. Many physical, engineering problems (the braking problem; the problem of contact; the problem of viscoelastic twisting), some economic problems (the problem of optimal constant basic capital; the problem of market equilibrium) and problems of computational geometry are often reduced to LCP ones. To the end of solving this problem we have applied the variational approach, which has given the possibility to reduce LCP to the problem of d.c. minimization. The results of testing the algorithm of global search in the problem stated have demonstrated an obvious high efficiency of the technique proposed. This result has been obtained on a sufficiently wide set of test examples of high dimension. We have solved nonconvex problems up to the dimension of 400 [13].

5.4. Problems of molecular biology and nanophysics

Consider the problem of minimization of the functional of complete energy in the model of charge transfer in a DNA molecule:

$$\left.\begin{aligned} F(x) = \langle x, Hx\rangle - \frac{1}{2}\sum_{i=1}^{n} k_i\,(x_i)^4 \downarrow \min_x, \\ x \in X \stackrel{\triangle}{=} \{x \in I\!R^n \mid \varphi(x) = 0\}\,, \end{aligned}\right\} \qquad (P)$$

where H is a 3-diagonal $(n \times n)$ matrix with nonnegative components, $k_i > 0 \;\; i = 1, \dots n$, and function $\varphi(x) \stackrel{\triangle}{=} \langle x,\; x\rangle - 1$. To the end of solving this problem we have used the strategy of global search in problems of d.c. minimization.

5.5. Problems of financial and medical diagnostics

Such problems are well-know as applied ones. These are often interpreted as the problems of generalized separability. For example, if the two sets of points $\mathcal{A}$ and $\mathcal{B}$ are characterized by the matrices $\mathcal{A} = [a^1, \dots, a^M],\; B = [b^1, \dots, b^N]$, $a^j, b^j \in I\!R^n$, then the problem of polyhedral separability may be reduced to the problem of minimization of the nonconvex nondifferentiable error function

$$F(V, \Gamma) = F_1(V, \Gamma) + F_2(V, \Gamma), \qquad (5.3)$$

$$\left.\begin{aligned} F_1(V, \Gamma) &= \frac{1}{M}\sum_{i=1}^{M} \max\{0, \max_{1\le p\le P}(\langle a^i, v^p\rangle - \gamma_p + 1)\}, \\ F_2(V, \Gamma) &= \frac{1}{N}\sum_{j=1}^{N} \max\{0; \min_{1\le p\le P}(-\langle b^j, v^p\rangle + \gamma_p + 1)\}. \end{aligned}\right\} \qquad (5.4)$$

Generalization of the theory of global search to be applied to the nonsmooth case has given the possibility to develop a software complex intended for solving nonsmooth problems of d.c. minimization [21]. This complex has proved to be rather efficient on a set of test examples characterized by high dimension (borrowed from the literature) and also on the examples constructed by the technique random generation [10].

5.6. Discrete programming

a) Problems of maximum clique and of maximum weighted clique

The maximum clique problem (MCP) implies the search for a maximally complete subgraph in a given non-oriented graph. In 1993, DIMACS Challenge related to this problem was conducted. Test examples as well as results of operation of the algorithms were published.

MCP conducted on the basis of the continuous Motskin-Strauss problem statement and the I.Bomze regularization, may be represented as a problem of maximization of the sign-definite quadratic form on the canonic symplex

$$\left.\begin{array}{c} F(x) = \sum\limits_{(i,j)\in E} x_i x_j = \langle x, Ax\rangle \uparrow \max, \\ x \in S_n = \{x = (x_1, ..., x_n)^T \mid \sum_{i=1}^n x_i = 1,\ x_i \geq 0,\ i = 1, .., n\}, \end{array}\right\}$$

where E is a set graph's edges; A is its contingency matrix.

We have proposed a different continuous statement of the MCP

$$\left.\begin{array}{c} \phi(x) \stackrel{\triangle}{=} \sum\limits_{i=1}^{n} x_i^2 \downarrow \min, \quad x \in S, \\ \Phi_0(x) \stackrel{\triangle}{=} \left\langle x, \overline{A}x \right\rangle \leq 0, \end{array}\right\}$$

where $\overline{A}$ is the contingency matrix of the complementarity graph, and the equivalence of this problem statement and the statement of MCP has been proved.

With the use of above problem statements we have developed the algorithms intended for solving MCP and the conducted computational experiments, which has led to the following conclusion. The approach we have proposed is competitive on many classes of text examples, and its further development and parallelization may allow us to confirm the success gained [12].

Generalization of the results obtained for MCP in order to apply these to MWCP has been conducted. The numerical experiment related to solving MWCP for the continuous problem statements on the weighted graphs from the DIMACS library has demonstrated high efficiency of the approach proposed.

b) **The problem of p-median and the problem of dislocation with preferences**

The problem of p-median is a well known NP-hard problem, which implies finding of p nodes (medians) of the weighted oriented graph to the end of minimization of the total distance from the nearest medians to non-median nodes of the graph.

Having introduced the binary variables y_i and x_{ij}), which correspond to the nodes $i \in V$ and to the edges $ij \in A$, we obtain the following problem of integer-numeric linear programming:

$$
\begin{gathered}
\sum_{ij \in A} w_{ij} x_{ij} \downarrow \min_{x,y}, \\
\sum_{j \in V, i \neq j} x_{ij} + y_i = 1 \quad \forall i \in V, \quad x_{ij} \leq y_i \quad \forall ij \in A, \ i \in V, \\
\sum_{i \in V} y_i = p, \ \ y_i, x_{ij} \in \{0, 1\} \quad \forall ij \in A, i \in V.
\end{gathered}
\tag{5.5}
$$

The given problem is the basic one for many problems of dislocation, and it is used in the cluster analysis, where the set of objects must be separated into p subsets of similar objects. The similarity of objects is expressed in terms of the weight of the corresponding graph's edge. As far as high-dimensional problems of "optimal dislocation" related to constructing of transport vehicles is concerned, we have elaborated an algorithm capable of solving problems defined on the graphs having more than $5 \cdot 10^4$ nodes and $5 \cdot 10^6$ edges.

Besides a simplest problem of p-median we also considered a more general model — the problem of p-median with the preferences of the customers. This problem possesses a natural bilevel structure, when a set of median nodes is chosen on the upper level to the end of minimization of the distance down to non-median ones, while on the lower level there takes place correlation of the non-median nodes to the medians chosen on the upper level by minimizing the function of preferences. The discrete models obtained may be applied in systems of machine vision, which are intended for testing the quality of goods manufactured.

c) **Problems of multi-dimensional knapsack**

As far as the known combinatorial problem of multi-dimensional knapsack, which is reduced to the continuous inverse-convex problem of the following form is concerned

$$
\left.
\begin{gathered}
\langle c, x \rangle \uparrow \max, \quad \langle a, x \rangle \leq \beta, \\
\| \, x - \frac{e}{2} \, \|^2 - \frac{n}{4} \geq 0, \\
x \in \Pi = \{ x \in I\!R^n \mid 0 \leq x_i \leq 1 \}.
\end{gathered}
\right\}
\tag{5.6}
$$

we have developed new methods of local search, algorithms of global search, which represent a combination of known approaches and procedures in discrete optimization, which follow from the GOC. These methods have proved their comparative efficiency in testing on the well-known problems from the library DIMACS related to discrete optimization.

6. Optimal control (OC)

We have also proposed necessary and sufficient GOCs [8, 24] for the non-convex OC problems of the form

$$(\mathcal{P}1): \qquad J(u) \downarrow \min_u, \ u \in \mathcal{U}, \tag{6.1}$$

$$J(u) = g_1(x(t_1)) - h_1(x(t_1)) + \int_T [g(x(t),t) - h(x(t),t) + f(u(t),t)]dt,$$

$$\mathcal{U} = \{u(\cdot) \in L^r_\infty(T) \mid u(t) \in U \subset I\!R^r \ \forall t \in T\}, \tag{6.2}$$

where the functions $g_1(x), h_1(x)$ and $x \to g(x,t), h(x,t), t \in T$ are convex and differentiable within, say, the linear control system

$$\dot{x}(t) = A(t)x(t) + B(u(t),t), \quad x(t_0) = x_0. \tag{6.3}$$

On this basis of the GOCs we are developing the theory of global search, in particular, methods of local and global search [9, 15, 16]. We have already conducted a series of numerical experiments with high-dimensional test OC problems, in which there may be, for example, 60000 processes, which satisfy Pontryagin's maximum principle and have only one global solution. Our approach has shown high efficiency for all the problems generated. In all the cases it was possible to find the globally optimal process by passing through only a limited number of stationary (PMP) processes.

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Advanced methods for solving nonconvex problems of optimization and optimal control

Abstract. This paper contains the review of the results obtained in the last years in the theory and numeric methods of the solution of nonconvex optimization problems and optimal control problems.
Keywords: nonconvex optimization, d.c. function, local search, global search

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